

G-FANO THREEFOLDS, II

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ABSTRACT. We classify Fano threefolds with only Gorenstein terminal singularities and Picard number greater than 1 satisfying an additional assumption that the G -invariant part of the Weil divisor class group is of rank 1 with respect to an action of some group G .

1. INTRODUCTION.

This work is a sequel to [Pro10].

1.1. Let X be a Fano threefold with at worst terminal Gorenstein singularities defined over a field \mathbb{k} of characteristic 0. Assume that a group G acts on $\bar{X} := X \otimes_{\mathbb{k}} \bar{\mathbb{k}}$, where $\bar{\mathbb{k}}$ is the algebraic closure of \mathbb{k} . Moreover, we assume that X , G and \mathbb{k} satisfy one of the following two conditions.

- (a) *Algebraic case.* G is the Galois group of $\bar{\mathbb{k}}$ over \mathbb{k} acting on \bar{X} through the second factor. The action of G on X is trivial.
- (b) *Geometric case.* The field \mathbb{k} is algebraically closed, G is a finite group, and the action of G on X is given by a homomorphism $G \rightarrow \text{Aut}_{\mathbb{k}}(X)$.

We say that X is a *G-Fano threefold* if X has at worst terminal Gorenstein singularities, $-K_X$ is ample, and

$$(1.1.1) \quad \text{Cl}(X)^G \text{ is a subgroup of rank 1 containing } -K_X.$$

where $\text{Cl}(X)$ is the Weil divisor class group of X . We refer to the introduction in [Pro09], [Pro10], and [Pro10b] for the motivation behind this definition.

In this paper we give a classification of one class of Gorenstein G -Fano threefolds. More general we assume that the group G acts only on the Picard lattice $\text{Pic}(X)$ in an appropriate way (so we do not assume that G acts on the variety itself).

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1.2. Theorem. *Let X be a smooth Fano threefold over an algebraically closed field of characteristic 0. Assume that $\rho(X) > 1$ and a finite group G acts on $\text{Pic}(X)$ preserving the intersection form and the class $c_1 = [-K_X]$. Furthermore, assume that $\text{Pic}(X)^G \simeq \mathbb{Z}$. Then X is one of the varieties in the table below.*

No.	$\rho(X)$	$-K_X^3$	X
(1.2.1)	2	12	Let $Z_6 \subset \mathbb{P}^8$ be the image of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and let $Y_6 \subset \mathbb{P}^9$ be the projective cone over Z_6 . Then X is an intersection of Y_6 with a hyperplane and a quadric
(1.2.2)	2	20	X is an intersection of three divisors of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$
(1.2.3)	2	28	Let $\sigma : Y' \rightarrow \mathbb{P}^5$ be the blowup with center a Veronese surface. Then X is an intersection of $D_1 \in \sigma^* \mathcal{O}_{\mathbb{P}^5}(1) $ and $D_2 \in \sigma^* \mathcal{O}_{\mathbb{P}^5}(2) - E $ where E is the exceptional divisor
(1.2.4)	2	48	$V_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$, a divisor of bidegree $(1, 1)$
(1.2.5)	3	12	X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a member of $ -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1} $
(1.2.6)	3	30	X is an intersection of divisors of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$
(1.2.7)	3	48	$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
(1.2.8)	4	24	$X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of multi-degree $(1, 1, 1, 1)$

1.2.9. Remark. It is easy to see that all the Fano varieties in the table, except for (1.2.1) and (1.2.5), are rational. Varieties (1.2.1) and (1.2.5) are not rational (see [AB92]).

Theorem 1.2 will be proved in Sect. 5. In Sect. §2 we give several examples. Sections 3 and 4 are preliminary. Finally, in Sect. 6 we investigate singular G -Fano threefolds with $\rho(X) > 1$.

Notation. We work over an algebraically closed field \mathbb{k} of characteristic 0. By $c_1 \in \text{Pic}(X)$ we denote the anticanonical class $-K_X$ of X , $\text{Cl}(X)$ denotes the Weil divisor class group.

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2. EXAMPLES AND REMARKS

2.1. Remark. By construction, in all cases except for (1.2.5) and (1.2.7) the variety X is a complete intersection of certain Cartier divisors in some higher-dimensional variety Y , where $Y = Y_6$ in the case (1.2.1), $Y = \mathbb{P}^3 \times \mathbb{P}^3$ in the case (1.2.2), $Y = Y'$ in the case (1.2.3), $Y = \mathbb{P}^2 \times \mathbb{P}^2$ in the case (1.2.4), $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ in the case (1.2.6), and $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the case (1.2.8). The embedding $X \subset Y$ is canonical. This means, in particular, that automorphisms of X are induced by those of Y .

Below we give some comments on the varieties in the table.

2.2. Case (1.2.1). The variety X can be one of the following two types:

- a) $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bidegree $(2, 2)$, or
- b) X is a double cover of $V_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ (see (1.2.4)), whose branch locus is a member of $| -K_{V_6} |$.

2.3. Case (1.2.2). The variety X is the blow-up of \mathbb{P}^3 along a curve of degree 6 and genus 3 which is an intersection of cubics. Indeed, two projections $f_i : X \rightarrow \mathbb{P}^3$ are blowups of some smooth curve $C \subset \mathbb{P}^3$. Easy computations show that C must satisfy the above conditions.

Conversely, let X be the blow-up of \mathbb{P}^3 along a curve of degree 6 and genus 3 which is an intersection of cubics. By [MM83, §5] on X there are two blowup structures $f_i : X \rightarrow \mathbb{P}^3$ of this type. These two contractions induce a map $f = f_1 \times f_2 : X \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ which must be finite and birational onto its image. Consider the composition $f' : X \xrightarrow{f} \mathbb{P}^3 \times \mathbb{P}^3 \xrightarrow{s} \mathbb{P}^{15}$, where s is the Segre embedding. Let $M_i := f_i^* \mathcal{O}_{\mathbb{P}^3}(1)$. Then $-K_X = M_1 + M_2$. By Riemann-Roch $\dim | -K_X | = 12$. Therefore, $f'(X)$ is contained into a subspace of codimension 3, i.e., $f'(X) \subset s(\mathbb{P}^3 \times \mathbb{P}^3) \cap \mathbb{P}^{12}$. Since $\deg f'(X) = 20 = \deg s(\mathbb{P}^3 \times \mathbb{P}^3) \cap \mathbb{P}^{12}$, we have $f(X) \simeq f'(X) = s(\mathbb{P}^3 \times \mathbb{P}^3) \cap \mathbb{P}^{12}$. By subadjunction, $K_X = f^* K_{f(X)} - B$, where B is an effective divisor defined by the conductor ideal. Since, $\omega_{f(X)} = \mathcal{O}_{f(X)}(-1, -1)$, we have $B = 0$. Therefore, $f(X)$ is normal and $f(X) \simeq X$.

2.3.1. Example (cf. [Kat87]). Let $(a_{i,j}), (b_{i,j}), (c_{i,j})$ be symmetric 4×4 -matrices and let $X \subset \mathbb{P}_{x_1, \dots, x_4}^3 \times \mathbb{P}_{y_1, \dots, y_4}^3$ is given by the equations

$$\sum a_{i,j} x_i y_j = \sum b_{i,j} x_i y_j = \sum c_{i,j} x_i y_j = 0.$$

If $(a_{i,j}), (b_{i,j}), (c_{i,j})$ are taken sufficiently general, then X is a smooth Fano threefold of type (1.2.2). The group μ_2 acts on X by $x_i \mapsto y_i$. The induced birational involution $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is cubo-cubic, it is given by the linear system of cubics passing through the center of blowup $C \subset \mathbb{P}^3$. The exceptional divisor is sweep out by trisecants of C and it is a surface $F \subset \mathbb{P}^3$ of degree 8 with multiplicity 3 along C [Kat87].

2.4. Case (1.2.3). It is easy to show that Y' is a Fano fivefold. Linear systems $|\mathcal{O}_{\mathbb{P}^5}(1)|$ and $|\mathcal{O}_{\mathbb{P}^5}(2) - E|$ define two contractions $h_i : Y' \rightarrow \mathbb{P}^5$ which are blowups of Veronese surfaces $V = V_4 \subset \mathbb{P}^5$ (cf. [CK89]). In this case, X can be realized as the blow-up of a smooth quadric $Q \subset \mathbb{P}^4$ along a twisted quartic curve. Indeed, the restriction $f_1 = h_1|_X$ is a birational map whose image is a quadric $Q = h_1(D_1) \cap h_2(D_2) \subset \mathbb{P}^5$ and moreover f_i is the blowup of Q along $V \cap h_1(D_1)$, a twisted quartic curve.

2.4.1. Example. Let $C \subset \mathbb{P}^4$ be a rational normal curve of degree 4. The action of the group $\text{Aut}(C) = \text{PGL}_2(\mathbb{k})$ naturally extends to \mathbb{P}^4 so that $\mathbb{P}^4 = \mathbb{P}(S^4 V)$, where V is the standard representation of $\text{GL}_2(\mathbb{k})$. The representation of $\text{GL}_2(\mathbb{k})$ on $S^4 V$ is irreducible and can be defined by matrices over \mathbb{R} . Therefore, there exists an invariant non-singular quadric $Q \subset \mathbb{P}^4$ containing C . Let $f_1 : X \rightarrow Q$ be the blowup of C . Then X is a Fano threefold of type (1.2.3). The $\text{PGL}_2(\mathbb{k})$ -action lifts to X and the second contraction $f_2 : X \rightarrow Q$ is also $\text{PGL}_2(\mathbb{k})$ -equivariant. Let $C' \subset Q$ be the center of the blowup f_2 . Clearly, the pairs (Q, C) and (Q, C') are $\text{SL}_2(\mathbb{k})$ -isomorphic, i.e. there exists an automorphism $\gamma : Q \rightarrow Q$ such that $\gamma(C) = C'$ (because C is the only one-dimensional orbit on Q). Then $f_2 \circ \gamma \circ f_1^{-1}$ is an involution on X .

2.5. Case (1.2.6). Here X is the blow-up of $V_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ (see (1.2.4)), along a curve C of bidegree $(2, 2)$ such that the composition $C \hookrightarrow V_6 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{p_i} \mathbb{P}^2$ is an embedding for each projection p_i , $i = 1, 2$. Indeed, there are three projections $\pi := X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$. The image of each π_i is contained into a divisor $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ of type $(1, 1)$ and π_i is birational onto its image. Hence π_i passes through a birational extremal contraction $X \rightarrow V' \rightarrow V$. By [MM82, Table 3] we have $V' \simeq V$, V is smooth and π_i is a blowup of a curve as above.

2.5.1. Example (cf. [Nak89]). Let $\Gamma \subset \mathbb{P}^2$ is a non-degenerate conic and let $\Gamma^* \subset \mathbb{P}^{2*}$ be its dual, the conic formed by lines that are tangent

to Γ . Consider the incidence curve

$$C = \{(P, L) \in \Gamma \times \Gamma^* \subset \mathbb{P}^2 \times \mathbb{P}^{2*} \mid L \text{ is tangent to } \Gamma \text{ at } P\}.$$

Then C is contained into the flag variety $\text{Fl}(\mathbb{P}^2) = V_6$ and satisfies conditions of (1.2.6). The action $\text{Aut}(\Gamma) = \text{PGL}_2(\mathbb{k})$ extends to $V_6 = \text{Fl}(\mathbb{P}^2)$. Orbits of $\text{PGL}_2(\mathbb{k})$ on V_6 are described as follows:

- $C \simeq \mathbb{P}^1 \simeq \text{SL}_2(\mathbb{k})/B$, where B is a Borel subgroup,
- D' and D'' , where $\bar{D}' := D' \cup C$ and $\bar{D}'' := D'' \cup C$ are complete surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the diagonal action of $\text{PGL}_2(\mathbb{k})$,
- an open orbit $U \simeq \text{SL}_2(\mathbb{k})/Q_8$, where Q_8 is the binary quaternion group of order 8.

Let $f : X \rightarrow V_6$ be the blowup of C . Then X is a Fano threefold of type (1.2.6) admitting a $\text{PGL}_2(\mathbb{k})$ -action. There are two more $\text{PGL}_2(\mathbb{k})$ -equivariant contractions $f', f'' : X \rightarrow V_6$ contracting proper transforms of \bar{D}' (resp. \bar{D}'') to curves $C' \subset V_6$ (resp. $C'' \subset V_6$) of bidegree $(2, 2)$. The pairs (V_6, C) , (V_6, C') , and (V_6, C'') , are isomorphic. These isomorphisms induce an action of the symmetric group \mathfrak{S}_3 on X .

2.6. Case (1.2.8). The variety X is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along an elliptic curve which is an intersection of two members of $|- \frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|$. Similar to 2.3.1 one can easily construct examples of symmetric varieties of this type (admitting a G -structure).

3. ACTION ON THE PICARD LATTICE.

3.1. Lemma. *Let $V = \mathbb{Q}^N$ and let $\Phi : G \hookrightarrow \text{GL}(V)$ be a faithful representation of a finite group G . Identify G with its image $\Phi(G) \subset \text{GL}(V)$ and assume that $V^G = 0$. Then there is a subgroup $G_0 \subset G$ such that $V^{G_0} = 0$ and*

$$|G_0| = \begin{cases} 2 & \text{if } N = 1, \\ 2 \text{ or } 3 & \text{if } N = 2, \\ 2, 4, \text{ or } 6 & \text{if } N = 3, \\ 5, 9, \text{ or divides } 3 \cdot 2^7 & \text{if } N = 4. \end{cases}$$

Proof. Cases $N \leq 2$ are trivial. Consider the case $N = 3$. According to Minkowski's bound (see e.g. [Ser07]) the order of any subgroup $G \subset \text{GL}_3(\mathbb{Q})$ divides 48. Assume that there is an element $\tau \in G$ such that $V^{\langle \tau \rangle} = 0$. Let $\mu(t)$ be the minimal polynomial of τ . Then $\deg \mu \leq 3$, μ is a product of cyclotomic polynomials $\phi_k(t)$, and $\mu(1) \neq 0$. Moreover, in some basis, τ is given by an orthogonal matrix, so $\mu(-1) = 0$ and $\phi_2 \mid \mu$. Hence, there are the following possibilities: $\mu = \phi_2, \phi_2\phi_4, \phi_2\phi_3$, or $\phi_2\phi_6$. Thus we can take $G_0 = \langle \tau \rangle$.

Now we assume that for any element $\tau \in G$ we have $V^{(\tau)} \neq 0$. In particular, G is not a cyclic group. We also may assume that $|G| \geq 8$. If G contains a subgroup $G_1 \subset G$ isomorphic to $\mu_2 \times \mu_2$, then either $G_1 \subset \mathrm{SL}(V)$ or $G_1 \ni -\mathrm{id}$. In the latter case we can take $G_0 = \langle -\mathrm{id} \rangle$ and in the former case we can take $G_0 = G_1$. Note that $\mathrm{GL}(V)$ contains no elements of order 8 and the quaternion group has no real faithful representations of dimension ≤ 3 . Thus we may assume that the order of G is not divisible by 8. We get only one possibility: Sylow 2-subgroups of G are cyclic of order 4 and G is of order 12. Let $G_1 \subset G$ be a Sylow 2-subgroup and let τ be its generator. We may assume that $G_1 \subset \mathrm{SL}(V)$ (otherwise we can take $G_0 = G_1$). Then $G \subset \mathrm{SL}(V)$. If G has an element ξ of order 6, then ξ generates the desired subgroup G_0 . Otherwise the representation $\Phi : G \hookrightarrow \mathrm{SL}(V)$ is irreducible. Then we must have $G \simeq \mathfrak{A}_4$ and G_1 is not cyclic, a contradiction.

Consider the case $N = 4$. Again according to Minkowski's bound (see e.g. [Ser07]) the order of any subgroup $G \subset \mathrm{GL}_4(\mathbb{Q})$ divides $2^7 \cdot 3^2 \cdot 5$. Again if there is an element $\tau \in G$ such that $V^{(\tau)} = 0$, then its minimal polynomial $\mu(t)$ satisfies the following conditions: $\deg \mu \leq 4$, μ is a product of cyclotomic polynomials $\phi_k(t)$, and $\mu(1) \neq 0$. In this case, we have either $\mu = \phi_5, \phi_8, \phi_{10}, \phi_{12}$, or μ divides $\phi_2 \phi_3 \phi_4 \phi_6$. Thus we can take $G_0 = \langle \tau \rangle$.

Now we assume that for any element $\tau \in G$ we have $V^{(\tau)} \neq 0$. Then G contains no elements of order 5, 9, and 8. In particular, the order of G divides $2^7 \cdot 3^2$. Let $G_1 \subset G$ be a Sylow 3-subgroup. By the above, we may assume that $G_1 \simeq \mu_3 \times \mu_3$. In this case $V^{G_1} = \{0\}$ and we can take $G_0 = G_1$. \square

3.1.1. Corollary. *In notation of Theorem 1.2 there is a subgroup $G_0 \subset G$ of order N such that $\rho(X)^{G_0} = 1$, where*

$$N = \begin{cases} 2 & \text{if } \rho(X) = 2, \\ 2 \text{ or } 3 & \text{if } \rho(X) = 3, \\ 2, 4 \text{ or } 6 & \text{if } \rho(X) = 4, \\ 5, 9, \text{ or divides } 3 \cdot 2^7 & \text{if } \rho(X) = 5. \end{cases}$$

Proof. Let V be the orthogonal complement to c_1 in $\mathrm{Pic}_{\mathbb{Q}}(X) := \mathrm{Pic}(X) \otimes \mathbb{Q}$. The faithful representation $G \hookrightarrow \mathrm{GL}(V)$ satisfies the condition $V^G = 0$. Then we can apply Lemma 3.1. \square

4. FANO CONIC BUNDLES AND DEL PEZZO FIBRATIONS.

4.1. Definition. Let X be a smooth threefold. A morphism $f : X \rightarrow Z$ onto a smooth surface is a *conic bundle* if every (scheme) fiber is isomorphic to a conic in \mathbb{P}^2 . A morphism $f : X \rightarrow Z$ onto a

smooth curve is a *del Pezzo bundle* if f has connected fibers and $-K_X$ is f -ample. (In this case a general fiber of f is a del Pezzo surface.)

4.1.1. Remark. Let X be a smooth Fano threefold and let $f : X \rightarrow Z$ be a surjective morphism with connected fibers.

- (i) If Z is a smooth curve, then f is a del Pezzo bundle and $Z \simeq \mathbb{P}^1$.
- (ii) If Z is a smooth surface and f is equidimensional, then f is a conic bundle and Z is rational.

4.1.2. Proposition ([MM86, §4]). *Let X be a Fano threefold. Assume that X has a conic bundle structure $f : X \rightarrow Z$. Then*

- (i) Z is a del Pezzo surface.
- (ii) The discriminant curve $\Delta \subset Z$ is a curve with at worst ordinary double points (or empty).
- (iii) $\rho(X/Z) = 1$ if and only if for any irreducible curve $C \subset Z$ its preimage $f^{-1}(C)$ is also irreducible.
- (iv) If $C \subset Z$ is an irreducible curve such that $f^{-1}(C)$ is reducible, then C is a smooth connected component of Δ .
- (v) $h^{1,2}(X) = \rho(X) - \rho(Z) + p_a(\Delta) - 2$.

4.1.3. Corollary. *If in the assumptions of Proposition 4.1.2 $\rho(Z) \geq 2$, then X has a del Pezzo bundle structure.*

4.2. Assumption. From now on we assume that X is a smooth Fano threefold satisfying assumptions of Theorem 1.2. By [Pro10] we may assume that the Fano index of X is equal to 1 (otherwise X is of type (1.2.4) or (1.2.7)). Thus from now on we assume that $\text{Pic}(X)^G$ is generated by $-K_X$. We also assume that G is the smallest group satisfying condition (1.1.1) (cf. Corollary 3.1.1).

4.3. Proposition. *Assume that X has a conic bundle structure $f : X \rightarrow Z$ over $Z = \mathbb{P}^2$. Then X is of type (1.2.1) or (1.2.6).*

Proof. According to [MM83, Prop. 6.3] $\rho(X) \leq 3$. By Corollary 3.1.1 we may assume that $|G| \leq 3$. Let $L \subset \mathbb{P}^2$ be a line, let $F := f^{-1}(L)$, and let $\Delta \subset \mathbb{P}^2$ be the discriminant curve. Let F_1, \dots, F_n be the G -orbit of the class of F in $\text{Pic}(X)$, where $n = 2$ or 3 . Write $\sum F_i = ac_1$ for some $a \in \mathbb{Z}$. Take the line L to be sufficiently general, so the surface F is smooth. Then we have

$$(4.3.1) \quad n(12 - \deg \Delta) = nK_F^2 + 4n = n(K_X + F)^2 \cdot F - 2nc_1 \cdot F^2 = nc_1^2 \cdot F = c_1^2 \cdot \sum F_i = ac_1^3 > 0.$$

In particular, $\deg \Delta < 12$.

First assume that $n = 2$. As above $F_1 + F_2 = ac_1$ and so

$$a^2 c_1^3 = \frac{1}{a}(F_1 + F_2)^3 = \frac{6}{a}F_1^2 \cdot F_2 = \frac{6}{a}F_1^2 \cdot (F_1 + F_2) = 6c_1 \cdot F^2 = 12.$$

Since c_1^3 is even, there is only one possibility: $a = 1$, $c_1^3 = 12$, and $\deg \Delta = 6$. By Proposition 4.1.2 $h^{1,2}(X) = \rho(X) + 7$. Then from tables in [MM82] we get the case (1.2.1).

Assume that $n = 3$. Then $\rho(X) = 3$ and $\Delta \neq \emptyset$. From (4.3.1) we see $3(12 - \deg \Delta) = ac_1^3$. So, $ac_1^3 \leq 33$.

If $a \geq 2$, then from [MM82, Table 3] we get $a = 2$. Then $c_1^3 = 18 - \frac{3}{2}\deg \Delta$. Since c_1^3 is even, $c_1^3 = 12$ and $\deg \Delta = 4$. But in this case, $h^{1,2}(X) = p_a(\Delta) = 3$. This contradicts [MM82, Table 3]. Therefore, $a = 1$ and $c_1^3 = 3(12 - \deg \Delta)$. In particular, $\deg \Delta$ is even and $c_1^3 \leq 30$. Moreover, if $c_1^3 = 30$, then by [MM82, Table 3] we get the case (1.2.6). Thus we may assume that $c_1^3 < 30$. There are the following possibilities:

- $\deg \Delta = 4$, $c_1^3 = 24$, $h^{1,2}(X) = p_a(\Delta) = 3$,
- $\deg \Delta = 6$, $c_1^3 = 18$, $h^{1,2}(X) = p_a(\Delta) = 10$,
- $\deg \Delta = 8$, $c_1^3 = 12$, $h^{1,2}(X) = p_a(\Delta) = 21$.

All these cases are impossible by the classification [MM82, Table 3]. \square

4.4. Lemma. *Assume that X has a del Pezzo bundle structure $f : X \rightarrow \mathbb{P}^1$ and let F be a general fiber. Let F_1, \dots, F_n be the G -orbit of the class of F in $\text{Pic}(X)$. Write $\sum F_i = ac_1$ for some $a \in \mathbb{Z}$. Then*

- (i) $9n \geq nK_F^2 = ac_1^3$, $a > 0$,
- (ii) $n > 2$, $|G| > 2$, and $\rho(X) \geq 3$,
- (iii) ac_1^3 is divisible by 3,
- (iv) aK_F^2 is even,
- (v) if $n = 3$, then $\rho(X) = 3$ and either
 - (a) X is of type (1.2.5), or
 - (b) $c_1^3 = 24$ and $K_F^2 = 8$.

Proof. We have

$$9n \geq nK_F^2 = nc_1^2 \cdot F = c_1^2 \cdot \sum F_i = ac_1^3.$$

This proves (i). Further,

$$(4.4.1) \quad 6 \sum_{i < j < k} F_i \cdot F_j \cdot F_k = \left(\sum F_i \right)^3 = a^3 c_1^3 > 0.$$

In particular, ac_1^3 is divisible by 3 and $n > 2$. Further,

$$2F \cdot \sum_{1 \leq i < j} F_i \cdot F_j = F \cdot \left(\sum F_i \right)^2 = a^2 F \cdot c_1^2 = a^2 K_F^2.$$

Hence, $a^2 K_F^2$ is even.

Finally, let $n = 3$. Then $\rho(X) \geq 3$. If $c_1^3 \leq 12$, then by [MM82, Tables 3-5] we are in the case (1.2.5). Thus we may assume that $c_1^3 > 12$. Since c_1^3 is even, $ac_1^3 \leq 24$. Therefore, $a = 1$. Let $\Gamma := F_2|_F$. Further,

$$-K_F \cdot \Gamma = c_1 \cdot F_2 \cdot F_1 = F_1 \cdot F_2 \cdot F_3 = \frac{1}{6}(F_1 + F_2 + F_3)^3 = \frac{c_1^3}{6}.$$

Since $\Gamma^2 = F_2^2 \cdot F = 0$, by Riemann-Roch $K_F \cdot \Gamma$ is even. Hence c_1^3 is divisible by 12. We get $c_1^3 = 24$ and $K_F^2 = 8$. If $\rho(X) \geq 4$, then by [MM82, Tables 4-5] we have $\rho(X) = 4$ and X is of type (1.2.8). But in this case, X has a del Pezzo bundle structure of degree 6, a contradiction. \square

4.5. Lemma. $X \not\cong Z \times \mathbb{P}^1$, where Z is a smooth surface.

Proof. Clearly, Z is a del Pezzo surface of degree $10 - \rho(Z)$. Let F be a fiber of the projection $X = Z \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Take an element $\tau \in G$ so that ${}^\tau F$ is not proportional to F . Then ${}^\tau F \sim \alpha F + f^*L$ for some $0 \neq L \in \text{Pic}(Z)$ and $\alpha \in \mathbb{Z}$. Since $F^2 \equiv 0$, we have

$$0 = {}^\tau F^2 \cdot F = f^*L^2 \cdot F.$$

Hence, $L^2 = 0$ and $2\alpha F \cdot f^*L \equiv {}^\tau F^2 \equiv 0$. So, $\alpha = 0$ and ${}^\tau F = f^*L$. Further, by Riemann-Roch $K_Z \cdot L$ is even and

$$K_Z^2 = c_1^2 \cdot F = c_1^2 \cdot {}^\tau F = c_1^2 \cdot f^*L = (2F - f^*K_Z)^2 \cdot f^*L = -4K_Z \cdot L.$$

Therefore, $K_Z^2 = 8$ and $\rho(X) = 3$. This contradicts Lemma 4.4. \square

5. PROOF OF THEOREM 1.2

Recall our assumption that $\text{Pic}(X)^G$ is generated by c_1 and that G is the smallest group satisfying condition (1.1.1).

5.1. Consider the case $\rho(X) = 2$. Denote the generator of $G \simeq \mu_2$ by τ . Let R_1 and R_2 be extremal rays of the Mori cone $\overline{\text{NE}}(X) \subset N_1(X) \simeq \mathbb{R}^2$. Let $f_i : X \rightarrow X_i$ be the contraction of R_i , let A_i be the ample generator of $\text{Pic}(X_i) \simeq \mathbb{Z}$, and let $M_i := f_i^*A_i$. By Proposition 4.3 and Lemma 4.4 we may assume that both contractions are birational. Let D_i be the exceptional divisor of f_i .

5.1.1. Theorem ([MM83, Th. 5.1]). *The group $\text{Pic}(X)$ is generated by M_1 and M_2 .*

By this theorem

$$-K_X \equiv \alpha_1 M_1 + \alpha_2 M_2, \quad \alpha_i \in \mathbb{Z}$$

Since $M_1 + {}^\tau M_1$ and $M_2 + {}^\tau M_2$ are invariant divisors, we have

$$M_1 + {}^\tau M_1 = -a_1 K_X, \quad M_2 + {}^\tau M_2 = -a_2 K_X,$$

for some $a_1, a_2 \in \mathbb{Z}$. Then

$$-2K_X = \alpha_1 M_1 + \alpha_2 M_2 + {}^\tau(\alpha_1 M_1 + \alpha_2 M_2) = -(\alpha_1 a_1 + \alpha_2 a_2) K_X.$$

So, $\alpha_1 a_1 + \alpha_2 a_2 = 2$. On the other hand,

$$0 < 2K_X^2 \cdot M_i = K_X^2 \cdot (M_i + {}^\tau M_i) = a_i c_1^3.$$

Therefore, $a_i > 0$. This gives us $a_i = \alpha_i = 1$, i.e.,

$$(5.1.2) \quad {}^\tau M_1 = M_2, \quad -K_X = M_1 + M_2.$$

Further, $M_1^2 \cdot D_1 = M_2^2 \cdot D_2 = 0$ and $M_2^2 \cdot {}^\tau D_1 = {}^\tau M_1^2 \cdot {}^\tau D_1 = 0$. Hence, D_2 and ${}^\tau D_1$ are proportional, so ${}^\tau D_1 = bD_2$ for some $b \in \mathbb{Q}$. Since $D_2 \cdot (-K_X)^2 > 0$ and ${}^\tau D_1 \cdot (-K_X)^2 > 0$, $b > 0$. Thus

$$D_1 + bD_2 = D_1 + {}^\tau D_1 = -cK_X \quad \text{for some } c \in \mathbb{Z}_{>0}.$$

If $f_1(D_1)$ is a point, then $M_1 \cdot D_1 \equiv M_2 \cdot D_2 \equiv 0$, i.e. $f_2(D_2)$ is also a point. In this case, $D_1 \cap D_2 = \emptyset$. Hence,

$$0 < c^2(-K_X)^3 = (-K_X) \cdot (D_1 + bD_2)^2 = (-K_X) \cdot D_1^2 + b^2(-K_X) \cdot D_2^2 < 0,$$

a contradiction.

Therefore, $C_i := f_i(D_i)$ are curves. In this case both varieties X_i are smooth Fano threefolds. Write $-K_{X_i} = r_i A_i$, $r_i = 1, 2, 3$, or 4 . Since D_2 and ${}^\tau D_1$ are primitive elements of $\text{Pic}(X)$, we have $D_2 = {}^\tau D_1$. Then

$$-K_X = -f^* K_{X_1} - D_1 = r_1 M_1 - D_1 = r_1 {}^\tau M_1 - {}^\tau D_1 = r_1 M_2 - D_2.$$

This shows that ${}^\tau D_1 = D_2$ and $r_1 = r_2$. Put $r := r_1 = r_2$. Further,

$$M_2^2 \cdot M_1 = M_1^2 \cdot M_2 = M_1^2 \cdot ((r-1)M_1 - D_1) = (r-1)M_1^3,$$

$$A_1 \cdot C_1 = -M_1 \cdot D_1^2 = -M_1 \cdot ((r-1)M_1 - M_2)^2 = (r-1)(r-2)M_1^3.$$

Therefore, $r \geq 3$. We get two possibilities: (1.2.2) and (1.2.3).

From now on we assume that $\rho(X) \geq 3$.

5.2. Proposition. *If in the above assumptions $\rho(X) \geq 3$ and X is not of type (1.2.6), then X has a structure of del Pezzo bundle of degree ≤ 8 .*

Proof. If X has a conic bundle structure $f : X \rightarrow Z$, then by Proposition 4.3 $Z \not\cong \mathbb{P}^2$. Since Z is a smooth rational surface, there exists a surjective morphism $Z \rightarrow \mathbb{P}^1$ with connected fibers. The composition map gives a structure of del Pezzo bundle whose fibers F are del Pezzos with $\rho(F) > 1$.

Assume that X has no conic bundle structures. Then by [MM83, (9.1), (9.2), (9.6)] the variety X is isomorphic to the blow-up of \mathbb{P}^3 along a disjoint union of a line and a conic. The map $X \rightarrow \mathbb{P}^3$ can be decomposed $X \rightarrow X_1 \rightarrow \mathbb{P}^3$, where $X_1 \rightarrow \mathbb{P}^3$ is the blow-up of a line. In this case X_1 has a structure of a \mathbb{P}^2 -fibration over \mathbb{P}^1 . The composition $X \rightarrow X_1 \rightarrow \mathbb{P}^1$ is the desired map. \square

5.3. Assume that $\rho(X) = 3$. By Proposition 5.2 X has a del Pezzo bundle structure $f : X \rightarrow Z$ and by Lemma 4.4 $n := |G| > 2$. Let F be a general fiber. By Lemma 4.4 we have $K_F^2 = 8$ and $c_1^3 = 24$. According to [MM82, Table 3] we have the following possibilities:

5.3.1. Case 7°. X is a blow-up of $V_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ along an elliptic curve C which is an intersection of two members of $|\frac{1}{2}K_{V_6}|$. Then projection from C gives us a del Pezzo bundle structure $f : X \rightarrow \mathbb{P}^1$ of degree 6. This contradicts $K_F^2 = 8$.

5.3.2. Case 8°. X is a member of the linear system $|p_1^* \sigma^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(2)|$ on $\mathbb{F}_1 \times \mathbb{P}^2$, where p_i is the projection and $\sigma : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the blowing up. The composition $X \xrightarrow{p_1} \mathbb{F}_1 \xrightarrow{q} \mathbb{P}^1$, where q is the \mathbb{P}^1 -ruling, is a del Pezzo bundle whose general fiber F is a divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(1, 2)$. Hence, F is a del Pezzo surface of degree 5. This contradicts $K_F^2 = 8$.

5.4. Finally, let X be a Fano threefold with $\rho(X) \geq 4$. By [MM82, Tables 4-5] $c_1^3 \geq 24$. Moreover, if $c_1^3 = 24$, then X is a divisor of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, i.e. X is of type (1.2.8). Thus from now on we assume that $c_1^3 \geq 26$ and $4 \leq \rho(X) \leq 5$ (see [MM82, Tables 4-5]).

5.5. Assume that $\rho(X) = 4$. Then by Corollary 3.1.1 $n = 2, 3, 4$, or 6. According to Lemma 4.4 (i)

$$(5.5.1) \quad 48 \geq nK_F^2 = ac_1^3$$

and ac_1^3 is divisible by 3.

Since, $c_1^3 \geq 26$, we have $a = 1$, $nK_F^2 \geq 26$, and $n = 4$ or 6. Further, by Lemma 4.4 (iv) K_F^2 is even. Hence, c_1^3 is divisible by 12. By [MM82, Table 4] we get the following possibility 7°: X is the blow-up of $V_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ of type (1.2.4) along disjoint union of curves of bidegree $(1, 0)$ and $(0, 1)$. So, there is an embedding $X \subset \mathbb{F}_1 \times \mathbb{F}_1$. The composition

$$X \hookrightarrow \mathbb{F}_1 \times \mathbb{F}_1 \xrightarrow{p_1} \mathbb{F}_1 \rightarrow \mathbb{P}^1$$

is a del Pezzo bundle whose general fiber is a del Pezzo surface of degree 7. This contradicts (5.5.1).

5.6. Assume that $\rho(X) = 5$. Then by [MM82, Table 5] and Lemma 4.5 we have two possibilities:

5.6.1. Case 1°. Then $c_1^3 = 28$ and X can be obtained as a sequence of blow-ups

$$X \xrightarrow{h} Y \xrightarrow{g} Q,$$

where Q is a smooth quadric in \mathbb{P}^4 , g is the blow-up of a conic $C \subset Q$, and h is the blow-up of three exceptional lines of g . As above, the projection $Q \dashrightarrow \mathbb{P}^1$ from C induces a quadric bundle $Y \rightarrow \mathbb{P}^1$. The composition $s : X \xrightarrow{h} Y \rightarrow \mathbb{P}^1$ is a del Pezzo bundle whose general fiber F is the blow-up of three points on the corresponding fiber of $Y \rightarrow \mathbb{P}^1$. Hence, F is a del Pezzo surface of degree 5. By Lemma 4.4 (i) we have $5n = nK_F^2 = 28a$. Since $|G_0|$ is not divisible by 7, we get a contradiction.

5.6.2. Case 2°. $c_1^3 = 36$ and X can be obtained as a sequence of blow-ups

$$X \xrightarrow{h} Y \xrightarrow{g} \mathbb{P}^3,$$

where g is the blow-up of two disjoint lines $L_1, L_2 \subset \mathbb{P}^3$ and h is the blow-up of two exceptional lines $\ell, \ell' \subset g^{-1}(L_1)$. The projection from L_2 induces a del Pezzo bundle $s : X \rightarrow \mathbb{P}^1$ of degree 8 and projection from L_1 induces a del Pezzo bundle $r : X \rightarrow \mathbb{P}^1$ of degree 6. Let F be a general fiber of s . Then by Lemma 4.4 (i)

$$8n = -aK_X^3 = 36a, \quad 2n = 9a.$$

Hence 9 divides n and we may assume that $n = 9$ and $a = 2$. By Lemma 3.1 and our assumption $|G| = 9$. Let F' be a general fiber of r . Again by Lemma 4.4 (i)

$$6n' = -a'K_X^3 = 36a', \quad n' = 6a'.$$

On the other hand, n' divides $|G| = 9$, a contradiction.

This finishes the proof of Theorem 1.2.

6. SINGULAR CASE

In this section we consider the case of singular G -Fano threefolds with $\rho(X) > 1$. On this step we may assume that the ground field is \mathbb{C} .

6.1. Theorem ([Nam97]). *Let X be a Fano threefold with terminal Gorenstein singularities. Then X is smoothable, that is, there exists a flat family $\mathfrak{X} \rightarrow \mathfrak{D} \ni 0$ over a small disc $(\mathfrak{D} \ni 0) \subset \mathbb{C}$ such that $\mathfrak{X}_0 \simeq X$ and a general member \mathfrak{X}_s , $s \in \mathfrak{D}$ is a smooth Fano threefold. Moreover, there is a natural identification $\text{Pic}(X) = \text{Pic}(\mathfrak{X}_s) = \text{Pic}(\mathfrak{X})$ so that $K_{\mathfrak{X}_s} = K_X$ (see [JR06b, §1]).*

6.2. Now let X be a Fano threefold with terminal Gorenstein singularities and let $\mathfrak{X} \rightarrow \mathfrak{D} \ni 0$ be its smoothing as above. We say that a reduced irreducible surface $S \subset X$ is a *plane* if $S \simeq \mathbb{P}^2$ and $\mathcal{O}_S(-K_X) = \mathcal{O}_{\mathbb{P}^2}(1)$. We say that a plane $S \subset X$ is *contractible* if there exists a birational morphism $f : X \rightarrow Y$ to a normal variety Y such that $\rho(X/Y) = 1$, S is contained into the exceptional locus $\text{Exc}(f)$ and S does not meet other components of $\text{Exc}(f)$ (if there is any). Note that, in this situation, $\text{Exc}(f)$ is of pure dimension 2 (cf. [Kac98, Prop. 1.4]).

Recall that the *nef cone* $\text{Nef}(X) \subset H^2(X, \mathbb{R})$ is the closed cone generated by nef divisors.

6.3. Proposition. *Assume that, in the above notation, X does not contain a contractible plane. Then under our identification $\text{Pic}(X) = \text{Pic}(\mathfrak{X}/\mathfrak{D}) = \text{Pic}(\mathfrak{X}_s)$, $s \in \mathfrak{D}$ we have $\text{Nef}(X) = \text{Nef}(\mathfrak{X}/\mathfrak{D}) = \text{Nef}(\mathfrak{X}_s)$.*

Proof. Note that \mathfrak{X} has only isolated hypersurface singularities. Hence, by [Gro68, Exp. XI, Corollary 3.14] the variety \mathfrak{X} is (locally) factorial. Let \mathcal{M} be a divisor on \mathfrak{X} , let $M := \mathcal{M}|_X$, and $\mathcal{M}_s := \mathcal{M}|_{\mathfrak{X}_s}$. If M is nef, then, obviously, so \mathcal{M}_s is. Thus it is sufficient to show the inverse implication. So, we assume that \mathcal{M}_s is nef for some $s \in \mathfrak{D}$ and M is not nef. Then \mathcal{M} is also not nef. Let

$$\lambda_0 := \sup\{\lambda \in \mathbb{Q} \mid \lambda\mathcal{M} - K_{\mathfrak{X}} \text{ is nef over } \mathfrak{D}\}.$$

By the rationality theorem we have $\lambda_0 \in \mathbb{Q}$ and there is an extremal ray R on $\mathfrak{X}/\mathfrak{D}$ such that $(\lambda_0\mathcal{M} - K_{\mathfrak{X}}) \cdot R = 0$ and $K_{\mathfrak{X}} \cdot R < 0$. Clearly, $\lambda_0\mathcal{M} - K_{\mathfrak{X}}$ is ample on \mathfrak{X}_s . Therefore, the locus $\text{Exc}(R)$ of the ray R does not meet \mathfrak{X}_s , so $\text{Exc}(R) \subset X$. In particular, this means that R is a flipping extremal ray. Then by [Kac98] the locus $\text{Exc}(R)$ is a disjoint union of irreducible surfaces S_i isomorphic to \mathbb{P}^2 and, moreover, $\mathcal{O}_{S_i}(-K_{\mathfrak{X}}) = \mathcal{O}_{\mathbb{P}^2}(1)$. Hence, X contains a contractible plane, a contradiction. \square

6.3.1. Corollary. *Notation as in 6.2. Assume that X does not contain a contractible plane. Let $\mathfrak{f}_s : \mathfrak{X}_s \rightarrow \mathfrak{Z}_s$ be an extremal contraction. Then there exists an extremal contraction $\mathfrak{f} : \mathfrak{X} \rightarrow \mathfrak{Z}$ over \mathfrak{D} such that the restriction $\mathfrak{f}|_{\mathfrak{X}_s}$ coincides with \mathfrak{f}_s .*

6.4. Proposition. *Let X be a G -Fano threefold. Then X does not contain a contractible plane.*

Proof. Assume the converse. Let $S \subset X$ be a contractible plane and let $f : X \rightarrow Y$ be its contraction as in 6.2. Let A be an ample Cartier divisor on Y and let $M := f^*A$. Thus M is a nef and big divisor on X and M is trivial on S . Now let $\{M_i\}$ and $\{S_{i,j}\}$ be G -orbits of M and S , respectively, where the subscript index i is chosen so that M_i

is trivial on $S_{i,j}$. Then M_i defines a contraction $f_i : X \rightarrow Y_i$ as in 6.2 and $\text{Exc}(f_i) = \cup_j S_{i,j}$. For each fixed i , the surfaces $S_{i,j}$ do not meet each other. Assume that the intersection $S_{i,j} \cap S_{i',j'}$ contains a curve C for some $i \neq i'$. Then $M_i \cdot C = M_{i'} \cdot C = 0$. So, C is contracted by f_i and $f_{i'}$. This contradicts $\rho(X/Y_i) = \rho(X/Y_{i'}) = 1$. Therefore, any two different planes from $\{S_{i,j}\}$ intersect each other by a set of dimension ≤ 0 . On the other hand, $\sum_{i,j} S_{i,j}$ is a Cartier divisor proportional to $-K_X$. In particular, $\sum_{i,j} S_{i,j}$ is connected and has only a local complete intersection singularities, a contradiction. \square

Let X be a G -Fano threefold with $\rho(X) > 1$ (and terminal Gorenstein singularities). According to Theorem 1.2 its smoothing is of one of the types (1.2.1) – (1.2.8). In this situation, we say that X is of the corresponding type (1.2.1) – (1.2.8).

6.5. Theorem. *Let X be a G -Fano threefold with $\rho(X) > 1$.*

- (i) *If X is of type (1.2.4) or (1.2.7), then X is smooth.*
- (ii) *If X is singular, then X has the same description as in the table of Theorem 1.2.*
- (iii) *If X is of type (1.2.3) and X is singular, then X is the blowup of a quadratic cone in \mathbb{P}^4 with center a union of two conics that do not pass through the vertex and meet each other transversely.*

Proof. (i) follows from [Pro10].

To prove (ii) we consider only the case (1.2.1). All other cases are similar (see 2.3 for the case (1.2.2)). Then, in notation of 6.2 and 6.3.1, $\rho(\mathfrak{X}/\mathfrak{D}) = 2$ and the nef cone $\text{Nef}(\mathfrak{X}/\mathfrak{D})$ has two edges. Thus there are two extremal contractions $\mathfrak{f}_i : \mathfrak{X} \rightarrow \mathfrak{Z}_i$, $i = 1, 2$ over $\mathfrak{D} \ni 0$. Let \mathcal{M}_i , $i = 1, 2$ be (integral) nef divisors generating edges of $\text{Nef}(\mathfrak{X}/\mathfrak{D})$. We can take \mathcal{M}_i to be primitive elements of $\text{Pic}(\mathfrak{X}/\mathfrak{D}) \simeq \mathbb{Z}^{\oplus 2}$. Thus $\mathcal{M}_i = \mathfrak{f}_i^* \mathcal{A}_i$, where \mathcal{A}_i is an ample generator of $\text{Pic}(\mathfrak{Z}_i/\mathfrak{D}) \simeq \mathbb{Z}$. By Corollary 6.3.1 the map \mathfrak{f}_i induces an extremal contraction $\mathfrak{f}_{i,s} : \mathfrak{X}_s \rightarrow \mathfrak{Z}_{i,s}$ on a each fiber. For a general fiber \mathfrak{X}_s , $s \neq 0$ we have $\mathfrak{Z}_{i,s} \simeq \mathbb{P}^2$. Let $f_i : X \rightarrow Z_i$ be the contraction induced on the central fiber $X = \mathfrak{X}_0$ and let $M_i := \mathcal{M}_i|_X$. Then $M_i = f_i^* A_i$, where A_i is an ample divisor on Z_i . By (5.1.2) $M_1 + M_2 = -K_X$. Since $M_i^3 = (\mathcal{M}_i|_{\mathfrak{X}_s})^3 = 0$, Z_i is a surface and f_i is a generically conic bundle. Hence, $A_i^2 = -\frac{1}{2} M_i^2 \cdot K_X = -(\mathcal{M}_i|_{\mathfrak{X}_s})^2 \cdot K_{\mathfrak{X}_s} = 1$. By semicontinuity $\dim H^0(Z_i, A_i) = \dim H^0(X, M_i) \geq \dim H^0(\mathfrak{X}_s, \mathcal{M}_i|_{\mathfrak{X}_s}) = 3$. Hence, by [Fuj75] $Z_i \simeq \mathbb{P}^2$. The map $f = f_1 \times f_2 : X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ must be finite and G -equivariant.

Let (d_1, d_2) be the bidegree of $f(X)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (as a divisor). Then

$$d_i = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0) \cdot \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0)^2 \cdot f(X) = \frac{M_1 \cdot M_2^2}{\deg f} = \frac{2}{\deg f}.$$

Similarly, $d_2 = 2/\deg f$. Thus $d_1 = d_2$ and there are two possibilities:

- a) $f(X)$ is of bidegree $(2, 2)$ and f is birational;
- b) $f(X)$ is of bidegree $(1, 1)$ and f is finite of degree 2.

In the first case, the map $f : X \rightarrow f(X)$ is birational and finite. By subadjunction $K_X = f^*K_{f(X)} - B$, where B is an effective divisor. On the other hand, $K_X = -M_1 - M_2 = f^*\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1) = f^*K_{f(X)}$. Hence, $B = 0$, $f(X)$ is normal, and $f : X \rightarrow f(X)$ is an isomorphism.

In the case b), we note that any irreducible singular hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ can be given, in some coordinate system, by the equation $x_1y_1 + x_2y_2 = 0$. In this case, the singular locus of $f(X)$ consists of one node and $f(X)$ contains exactly two planes. Therefore, $\text{rk Cl}(f(X))^G > 1$ and so $\text{rk Cl}(X) > 1$, a contradiction. Hence $f(X)$ is smooth.

Now we prove (iii). As above, by Corollary 6.3.1 two extremal contractions $\mathfrak{f}_i : \mathfrak{X} \rightarrow \mathfrak{Z}_i$, $i = 1, 2$ over $\mathfrak{D} \ni 0$ induce extremal contractions $\mathfrak{f}_{i,s} : \mathfrak{X}_s \rightarrow \mathfrak{Z}_{i,s}$ on a each fiber and for $s \neq 0$ the variety $\mathfrak{Z}_{i,s}$ is a smooth quadric. Hence \mathfrak{f}_i induces a birational extremal contraction $f_i : X \rightarrow Q_i$ on the central fiber $X = \mathfrak{X}_0$. Let $\mathcal{M}_i = \mathfrak{f}_i^*\mathcal{A}_i$, where \mathcal{A}_i is an ample generator of $\text{Pic}(\mathfrak{Z}_i/\mathfrak{D}) \simeq \mathbb{Z}$ and let $M_i := \mathcal{M}_i|_X$. Then $M_i = f_i^*A_i$, where A_i is an ample divisor on Z_i . By semicontinuity $\dim H^0(Q_i, A_i) \geq 5$. Hence, by [Fuj75] Q_i is a quadric in \mathbb{P}^4 . In particular, K_{Q_i} is Cartier. Since $-K_X$ is ample, by standard facts of the minimal model program Q_i has at worst terminal singularities. Moreover, f_i is an isomorphism over the singular locus of Q_i . Hence Q_i is either a smooth quadric or a quadratic cone in \mathbb{P}^4 . Let $\mathcal{E}_i \subset \mathfrak{X}$ be \mathfrak{f}_i -exceptional divisor. Since \mathfrak{X} is factorial, \mathcal{E}_i is an irreducible Cartier divisor. Let $E_i := \mathcal{E}_i|_X$. Clearly, the divisor E_i coincides with the f_i -exceptional locus. We have $-K_X = M_1 + M_2 = E_1 + E_2$ (because this holds on a general fiber).

Let $G_\bullet \subset G$ be the subgroup of index 2 that stabilizes M_1 (and M_2). Then the contraction f_i is G_\bullet -equivariant. Since $\text{rk Cl}(Q_i) \leq 2$, we have $\text{Cl}(Q_i)^{G_\bullet} \simeq \mathbb{Z}$.

Write $E_i = \sum_{j=1}^r k_i E_i^{(j)}$, where $k_i > 0$, $r \geq 1$, and $E_i^{(j)}$ are prime Weil divisors. Let $\Gamma_i := f_i(E_i)$ and let $\Gamma_i^{(j)} = f_i(E_i^{(j)})$. Since $\text{Cl}(X)^G$ is generated by K_X , the divisors $E_i^{(j)}$ form one G -orbit and $k_i = 1$ for all i . In particular, E_1 and E_2 have no common components and $\Gamma_i^{(j)}$ is a curve for all j (because $E_1 \neq E_2$ and $E_1 \cdot E_2 \cdot M_i \neq 0$). Let $D \in |-K_X|$ be a general member and let $D_i := f_i(D)$. Then $D_i \in |-K_{Q_i}|$. Since

the linear system $| -K_X |$ is base point free [JR06a], D is a smooth K3 surface. We may take S so that it does not contain non-trivial fibers of $E_i \rightarrow \Gamma_i$ and meets a general fiber transversely at one point. Then $D \rightarrow D_i$ is a finite birational morphism and D_i has at worst isolated singularities. Since D_i is Cohen-Macaulay, it is smooth and $D \rightarrow D_i$ is an isomorphism. Then

$$\deg \Gamma_i = A_i \cdot \Gamma_i = -K_X \cdot M_i \cdot E_i = 4,$$

$$-2 = -K_X \cdot E_i^2 = D \cdot E_i^2 = \Gamma_i^2 = 2p_a(\Gamma_i) - 2 \implies p_a(\Gamma_i) = 0.$$

Since $(-K_X)^2 \cdot E_1 = \frac{1}{2}(-K_X)^3 = 14$, r divides 14. Further, since $-K_X \cdot M_1 \cdot E_1 = 4$, r divides 4.

Therefore, $r = 1$ or 2 . If $r = 1$, i.e., E_i is irreducible, then so $\Gamma_i := f_i(E_i)$ is. Thus Γ_i is a smooth rational curve of degree 4. In this case, Γ_i is contained into the smooth locus of Q_i , f_i is the blowup of Γ_i , and X is smooth along E_i [Cut88]. Assume that Q_i is singular, i.e., it is the projective cone over a smooth quadric $W \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^3 . Let $O_i \in Q_i$ be the vertex of the cone. Then X has a unique singular point $O \in X$ and $f_i(O) = O_i$. Moreover, $O \notin E_i$. Since $E_1 + E_2 \sim -K_X$, $f_1(E_2) \sim -K_{Q_1}$. Moreover, $f_1(E_2)$ is singular along Γ_1 . Hence every 2-secant line to Γ_1 is contained in $f_1(E_2)$. Since $O_1 \notin f_1(E_2)$, the projection of Γ_1 from O_1 to W is an isomorphism. Thus Γ_1 is contained into a divisor of type $(1, 3)$ on Q_1 . But in this case, the action of G_\bullet on $\text{Cl}(Q_i) \simeq \mathbb{Z} \oplus \mathbb{Z}$ must be trivial, a contradiction.

Finally, assume that $r = 2$, i.e., $E_i = E_i^{(1)} + E_i^{(2)}$. Since the divisors $E_i^{(1)}$ and $E_i^{(2)}$ are not Cartier, G_\bullet interchanges $E_i^{(1)}$ and $E_i^{(2)}$. Hence, G_\bullet interchanges also $f_1(E_2^{(1)})$ and $f_1(E_2^{(2)})$. On the other hand, $f_1(E_2^{(1)}) + f_1(E_2^{(2)}) \sim -K_{Q_1} \sim \mathcal{O}_{Q_1}(3)$. This is possible only if $f_1(E_2^{(1)})$ and $f_1(E_2^{(2)})$ are not Cartier. So, Q_1 is the quadratic cone and $f_1(E_2)$ contains its vertex. Again by [Cut88] X is the blowup of Q_1 along Γ_1 . \square

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